Eur. Phys. J. A **25**, 249–256 (2005)
DOI 10.1140/epja/i2005-10092-4

THE EUROPEAN PHYSICAL JOURNAL A

Regularization of singular terms in the $N\overline{N}$ potential model

O.D. Dalkarov and A.Yu. Voronin[a]

P.N. Lebedev Physical Institute, 53 Leninsky pr., 117924 Moscow, Russia

Received: 15 March 2005 / Revised version: 12 May 2005 /
Published online: 29 August 2005 – © Società Italiana di Fisica / Springer-Verlag 2005
Communicated by Th. Walcher

Abstract. We suggest a method of singular terms regularization in a potential model of the $N\overline{N}$ interaction. This method is free from uncertainties related to the usual cut-off procedure and is based on the fact that, in the presence of sufficiently strong short-range annihilation, N and $\overline{N}$ never approach close enough to each other. In such a case the low-energy scattering is shown to be fully determined by the OBEP tail, while any details of the short-range core of the $N\overline{N}$ interaction are excluded from the observables. The obtained results for S- and P-wave scattering lengths are in agreement with the well-established theoretical models.

PACS. 13.75.Cs Nucleon-nucleon interactions (including antinucleons, deuterons, etc.) – 21.30.-x Nuclear forces – 21.30.Fe Forces in hadronic systems and effective interactions – 24.10.Ht Optical and diffraction models

1 Introduction

During the last decades numerous nonrelativistic models of the $N\overline{N}$ low-energy interaction [1–10] have been suggested. The intriguing problem of the possible existence of the so-called quasi-nuclear $N\overline{N}$ states [2,11] has strongly stimulated the mentioned researches. The physical arguments in favor of such states are the following. The interaction between N and $\overline{N}$ is expected to be more attractive than the NN interaction, as it follows from the procedure of G-conjugation [1]. Such a strong attraction should produce a spectrum of $N\overline{N}$ quasi-bound states. At the same time the range of annihilation estimated from the position of the singularity nearest to the threshold in the Feynman annihilation diagrams turns out to be much smaller than the range of the meson exchange forces. This means that quasi-nuclear states could be narrow enough to be observed experimentally. It was indeed discovered in the Low-Energy Antiproton Ring (LEAR) experiments [12–14] that certain partial cross-sections sharply increase with energy decreasing down to the $N\overline{N}$ threshold (the so-called P-wave enhancement) which could be a manifestation of a narrow weakly bound state or resonance. This conclusion was verified by experiments with antiprotonic atoms [15,16] and by experimental studies of antiproton annihilation at rest by the OBELIX Collaboration [17–20]. The detailed review of mentioned issues can be found in [21,22]. Recent experimental data [23] on J/ψ decay into $\gamma p\bar{p}$ channel also indicate a strong enhancement near the $p\bar{p}$ threshold.

However, the transparent physical picture of the quasi-nuclear states has a significant drawback. The procedure of G-conjugation yields in attractive terms in $N\overline{N}$ potential of the type $-1/r^3$. It is well known, that attractive inverse power potentials $-1/r^s$ with $s > 2$ are singular [24], *i.e.* the scattering problem with standard (zero) boundary condition in the origin has no definite solution, while the spectrum of the system is not bounded from below. The usual way out is to impose that the singular behavior at short distances is an artifact of certain approximations (for instance the approximation of point-like nucleons, nonrelativistic approximation, etc). In the absence of the self-consistent theory it is common practice to introduce the phenomenological cut-off radius to regularize the singular behavior of the model at short distance. However the low-energy scattering observables change dramatically with small variations of the cut-off radius (as long as we deal with a real part of the $N\overline{N}$ potential) [25] and depend on the details of the cut-off procedure, which seriously diminish the predictive power of the model. In fact it is not clear if the quasi-nuclear near-threshold states are determined by the "physical part" of the OBEP, or they are artifacts, produced by the "non-physical" singular part of the interaction.

We suggest a model of $N\overline{N}$ interaction which is free from the above-mentioned uncertainties of the cut-off procedure. We will show that the strong inelastic interaction makes the low-energy scattering observables independent of any details of the short-range core, as far as the particle totally annihilates before it "falls to the center".

[a] e-mail: alexei.voronin@excite.com

The mathematical basis of our approach is the statement that an attractive inverse power potential of the type $-\gamma_s/r^s$ with $s \geq 2$ enables a unique solution of the Schrödinger equation with the standard boundary condition when it gets an *imaginary* addition to the interaction strength [26]. Curiously this is true even if such an addition is infinitesimal. It is shown that the scattering on the singular potential regularized in such a way is equivalent to the full absorption of the particles in the scattering center. Encouraged by the early result of Dalkarov and Myhrer [3], who introduced a full absorption boundary condition *at a certain inter-baryonic distance* and successfully described the low-energy $N\overline{N}$ scattering data, we suggest a regular potential model of the $N\overline{N}$ interaction, based on the OBE potential approach, but *without any cut-off*.

The important feature of our approach is that the reflected wave is generated only by the medium and the long-range part of the OBE potential, while part of the incoming flux which penetrates to smaller distances is fully absorbed. This means that any information about the low-energy $N\overline{N}$ scattering is determined by medium- and long-range parts of the OBEP only. We show that this property of the model is closely related to the phenomenon of the reflection of a quantum particle from the fast-changing attractive potential (the so-called quantum reflection).

We will demonstrate that the "singular" part of the potential when modified by annihilation cannot produce by itself any quasi-bound state. The known near-threshold resonances, which are well reproduced by our model, are determined by the long-range part of the OBEP. We calculate the scattering lengths in S and P partial waves for different values of spin, isospin and total momentum and demonstrate that the obtained results are in good agreement with the well-established theoretical models [7,8].

The paper is organized as follows. In the second section we discuss the general properties of the attractive inverse power potentials, regularized by an imaginary addition to the interaction strength. The third section is devoted to the construction of the regularized $N\overline{N}$ potential model.

2 Inverse power potentials with complex strength

In this section we present main results concerning the properties of inverse power potentials $-\gamma_s/r^s$ ($s \geq 2$) with complex strength $\gamma_s = \alpha_s + i\omega$. In the following we put the mass M of the particle equal to 1/2. Let us first treat the case $s > 2$. Near the origin we can neglect all the terms of the Schrödinger equation, increasing slower than $1/r^2$. We get the following expression for the wave function [27]:

$$\Phi(r) = \sqrt{r}\left(H_\mu^{(1)}(z) + \exp(2i\delta_0)H_\mu^{(2)}(z)\right), \tag{1}$$

$$z = \frac{2\sqrt{\gamma_s}}{s-2} r^{-(s-2)/2}, \tag{2}$$

$$\mu = \frac{2l+1}{s-2}, \tag{3}$$

Here $H_\mu^{(1)}(z)$ and $H_\mu^{(2)}(z)$ are the Hankel functions of order μ [28], l is the angular momentum, δ_0 is a phase-shift to be determined from the boundary condition. It is worth mentioning that the variable z is a semiclassical phase.

As long as the interaction strength γ_s is real ($\omega = 0$), both independent solutions have the same order of singularity in the origin and there is no obvious way to choose their linear combination. Thus zero boundary condition for the wave function in the origin does not supply us with a unique solution of the Schrödinger equation. The situation however changes when γ obtains an imaginary addition $\omega \neq 0$. Indeed, for $\omega > 0$ the only physical solution is $\sqrt{r}H_\mu^{(1)}(z)$, which is regular in the origin, while the second solution $\sqrt{r}H_\mu^{(2)}(z)$ exponentially diverges in the origin. (For $\omega < 0$ the physical solution is $\sqrt{r}H_\mu^{(2)}(z)$.) So far in the case of $\omega \neq 0$ the standard boundary condition selects the unique physical solution. In the following we will show that this statement is true even for infinitesimal values of ω.

To demonstrate explicitly the reason for the different behavior of the independent solutions in the origin, we replace the inverse power potential at distance less than r_0 by the constant $-\gamma_s/r_0^s$, having in mind to make r_0 tend to 0. Matching the logarithmic derivatives for the "square-well" solution and the solution (1) at small r_0, one can get for δ_0:

$$\delta_0 = \frac{s}{s-2} p(r_0) r_0, \tag{4}$$

$$p(r_0) = \frac{\sqrt{\alpha_s + i\omega}}{r_0^{s/2}}. \tag{5}$$

Now it is important that the interaction strength γ_s is complex. In the limit $r_0 \to 0$ we obtain

$$\lim_{r_0\to 0} \operatorname{Im}\delta_0 = \frac{s}{s-2}\operatorname{Im}\frac{\sqrt{\alpha_s+i\omega}}{r_0^{(s-2)/2}} \to \begin{cases} +\infty \text{ if } \omega > 0, \\ -\infty \text{ if } \omega < 0, \end{cases} \tag{6}$$

which means, that $\exp(2i\delta_0)$ is either 0 or ∞ and the linear combination of the Schrödinger equation solutions (1) is uniquely defined in the limit of zero cut-off radius r_0:

$$\lim_{r_0\to 0} \Phi(r) = \begin{cases} \sqrt{r}H_\mu^{(1)}(z) \text{ if } \omega > 0, \\ \sqrt{r}H_\mu^{(2)}(z) \text{ if } \omega < 0. \end{cases} \tag{7}$$

The above-described procedure clearly shows that the reason for the different behavior of two solutions in the origin comes from the fast increase of the potential, so that the imaginary part of the phase-shift δ_0 becomes infinite (6). Let us note here that vanishing of one of the solutions and divergence of the other in the origin is an intrinsic property of the Hankel functions $H_\mu^{(1,2)}(z)$ of large complex argument. This property is independent of the details of the cut-off procedure, which we used here only for the sake of clarity.

From the asymptotic expansion of the Hankel function of big argument (small r),

$$H_\mu^{(1,2)}(z) \simeq \sqrt{\frac{2}{\pi z}} \exp\left(\pm(iz - i\pi/4 - i\mu\pi/2)\right),$$

one can see that $\omega > 0$ selects an incoming wave , which corresponds to the full absorption of the particle in the scattering center, while $\omega < 0$ selects an outgoing wave, which corresponds to the creation of the particle in the scattering center.

As follows from (4) and (5) the above-mentioned conclusions are valid for any ω, including its infinitesimal value (as long as $s > 2$). It means, that the sign of an infinitesimal imaginary addition to the interaction constant selects the full absorption or full creation boundary condition (7). This boundary condition can be formulated as condition for the logarithmic derivative:

$$\lim_{r\to 0}\frac{\Phi'(r)}{\Phi(r)} = -i\,\mathrm{sign}(\omega)p(r)\,, \tag{8}$$

where $p(r)$ is the classical local momentum (5). (Compare with the boundary condition for incoming (outgoing) plane wave $\exp(\mp ipr)'/\exp(\mp ipr) = \mp ip$.)

As soon as the solution of the Schrödinger equation is uniquely defined, we can calculate the scattering observables. In particular we can now obtain the S-wave scattering length for the potential $-(\alpha_s \pm i0)/r^s$ (for $s > 3$):

$$a = \exp(\mp i\pi/(s-2))\left(\frac{\sqrt{\alpha_s}}{s-2}\right)^{2/(s-2)}\frac{\Gamma((s-3)/(s-2))}{\Gamma((s-1)/(s-2))}\,. \tag{9}$$

The fact that, in spite of $\mathrm{Im}\,\gamma_s \to \pm 0$, the scattering length has nonzero imaginary part is the manifestation of the singular properties of the mentioned potential which violates the self-adjointness of the Hamiltonian.

Let us compare the scattering length (9) with that of the repulsive inverse power potential α_s/r^s. One can get

$$a^{rep} = \left(\frac{\sqrt{\alpha_s}}{s-2}\right)^{2/(s-2)}\frac{\Gamma((s-3)/(s-2))}{\Gamma((s-1)/(s-2))}\,. \tag{10}$$

It is easy to see, that (9) can be obtained from (10) simply by choosing the certain branch of the function $\left(\sqrt{\alpha_s}\right)^{2/(s-2)}$ when passing through the branching point $\alpha_s = 0$. These branches correspond either to full absorbtion or creation of the particles in the scattering center. The scattering length in the regularized inverse power potential becomes an analytical function of γ_s in the whole complex plane of γ_s with a cut along the positive real axis. More generally, the low-energy scattering observables are uniquely defined for the inverse power potential with $s > 2$ when the strength parameter γ_s lies in the complex plane with a cut along the positive real axis.

Note that the boundary condition (7) of the full absorption (creation) is incompatible with the existence of any bound state. Indeed, one needs both the incoming and the reflected wave to form a standing wave, corresponding to a bound state. This means that the regularized inverse power potential supports *no bound states.* This is also clear from the above-mentioned fact that the scattering length for a regularized attractive inverse power potential is an analytical continuation of the scattering length of the repulsive potential.

Let us now turn to the important case of the inverse square potential $-\gamma_2/r^2$ with $\gamma_2 = \alpha_2 + i\omega$. The wave function now is

$$\Phi = \sqrt{r}\left[J_{\nu_+}(kr) + \exp(2i\delta_0)J_{\nu_-}(kr)\right], \tag{11}$$

$$\nu_\pm = \pm\sqrt{1/4 - \gamma_2}\,, \tag{12}$$

where k is the momentum of the quantum particle, and $J_{\nu_\pm}$ are the Bessel functions [28]. For simplicity we treat the case of zero angular momentum $l = 0$, the extension of the following results to the higher partial waves is straightforward. In the following we will be interested in the values of α_2 greater than critical, namely for $l = 0$ we consider $\alpha_2 > 1/4$. We use the same procedure as treated before of substituting the inverse power potential by a constant value at small r_0. Matching the logarithmic derivatives at r_0 we get for $\exp(2i\delta_0)$:

$$\lim_{r_0\to 0}\exp(2i\delta_0) \sim r_0^{\omega/\sqrt{\alpha_2 - 1/4}}\, r_0^{-2i\sqrt{\alpha_2-1/4}}\,.$$

It is clear, that due to the presence of imaginary addition ω we get $\mathrm{Im}\,\delta_0 \to \pm\infty$ when $r_0 \to 0$.

So far we come to the boundary condition

$$\lim_{r_0\to 0}\Phi(r) = \begin{cases}\sqrt{r}J_{\nu_+}(kr) \text{ if } \omega > 0\,,\\ \sqrt{r}J_{\nu_-}(kr) \text{ if } \omega < 0\,.\end{cases} \tag{13}$$

For the large argument this function behaves like

$$\Phi \sim \cos(z - \nu_\pm\pi/2 - \pi/4)\,.$$

In the limit $\omega \to \pm 0$ the corresponding scattering phase is

$$\delta = i\frac{\mathrm{sign}(\omega)\pi}{2}\sqrt{\alpha_2 - 1/4} + \pi/4\,. \tag{14}$$

As one can see, the S-matrix $S = \exp(2i\delta)$ is energy independent. This means that regularized inverse square potential supports *no bound states.* The regularized wave function and phase-shift are analytical functions of γ_2 in the whole complex plane with a cut $\alpha_2 > 1/4$.

It is worth mentioning that in the above regularization procedure we let the Hamiltonian be non–self-adjoint. This is a warrantable extension, as long as we are interested in the problems where the loss of particles from the initial channel is possible.

2.1 Regularization of a singular potential by a potential of inferior order

Now we would like to determine if it is possible to regularize a real attractive inverse power potential of given order s by an imaginary potential, which behaves in the origin less singular than $1/r^s$. In other words we would like to find the minimum power t_{min} of an *infinitesimal* imaginary inverse power potential required for the regularization of a given singular potential. The potential of interest is a sum $-\alpha_s/r^s - i\omega/r^t$. From expression (6) one immediately comes to the conclusion that the regularization is possible only if

$$t > s/2 + 1\,.$$

So far for $s > 2$ the regularization of a given singular inverse power potential $-\alpha_s/r^s$ is possible by adding the potential $i\omega/r^t$ with $\omega \to 0$. This imaginary potential may increase *slower* in the origin than the real one, as far as $t_{min} < s$ for $s > 2$.

This result establishes how "strong" should be the annihilation in the origin to ensure the full absorption of the particles in the presence of an attractive inverse power potential.

2.2 Quantum reflection from the inverse power potential

The phenomenon of the quantum reflection, *i.e.* over-barrier reflection from the fast-changing attractive potential is an old-known quantum-mechanical effect [29,30]. Such a phenomenon plays a crucial role in the low-energy scattering on regularized inverse power potential with an absorptive core. Indeed, the particles, which are not reflected from the fast-changing part of the potential are totally lost in its core. So far the scattering observables can bring information only about the characteristic distance where quantum reflection occurs. Such a characteristic distance would play a role of effective "annihilation" radius for the scattering on regularized inverse power potential with an absorbing core. It can be shown, that the phenomenon of quantum reflection is closely connected to the failure at certain distances of the WKB approximation applied to the low-energy scattering on the potential of interest [30,31].

The WKB approximation holds if $|\frac{\partial(1/p)}{\partial r}| \ll 1$, where $p(r)$ is the local classical momentum. In case of zero-energy scattering on regularized inverse power potential with $s > 2$ this condition is valid for

$$r \ll r_{sc} \equiv (2\sqrt{\alpha_s}/s)^{2/(s-2)}\,.$$

(For $s = 2$ the semiclassical approximation is valid only for $\alpha_2 \gg 1$.) The WKB approximation, consistent with the boundary condition (7) for $s > 2$ is

$$\Phi = \frac{1}{\sqrt{p(r)}} \exp\left(i\,\mathrm{sign}(\omega) \int_r^a p(r)\mathrm{d}r \right) \qquad (15)$$

with $p(r)$ from (5). This solution becomes accurate in the limit $r \to 0$. It follows from the above expression, that in case the WKB approximation is valid *everywhere*, the solution of the Schrödinger equation includes the incoming wave only (for distinctness we speak here of absorptive potential $\omega > 0$). The corresponding S-matrix is equal to zero, $S = 0$, within such an approximation. It is very important that this result is independent of any details of the inner part of the potential $p^2(r)$. It means that any modification of potential at $r \ll r_{sc}$ which keeps the validity of the WKB approximation (15) does not change the scattering observables.

The reflected wave can appear in the solution only in the regions where (15) does not hold. Namely, the particles are reflected from those parts of the potential which change sufficiently fast in comparison with the effective wavelength $|\frac{\partial(1/p)}{\partial r}| \geq 1$.

For the zero energy scattering and $l = 0$ this holds for $r \geq r_{sc}$.

The reflection coefficient $P \equiv |S|^2$, which shows the reflected part of the flux has the following form in the low-energy limit:

$$P = 1 - 4k|\,\mathrm{Im}\,a|,$$

where $k = \sqrt{E}$ is the momentum of the incident particle with energy E. This expression clearly shows, that for such slow particles that $k|\,\mathrm{Im}\,a| \ll 1$, the quantum reflection probability is close to unity. In the opposite limit of high energies $E \gg E_{sc} \equiv \alpha_s/r_{sc}^s = (s/2)^{2s/(s-2)}\alpha_s^{-2/(s-2)}$ the WKB holds everywhere and the S-wave reflection becomes exponentially small.

An important conclusion is that any information, which comes from the scattering on the absorptive singular potential is due to the quantum reflection. The distance where WKB approximation fails plays a role of effective "annihilation" radius. The low-energy scattering observables are sensitive to the potential details above this radius and practically independent of any "smooth" modification (*i.e.* modification which does not violate the WKB approximation) of the potential below this radius.

2.3 Near-threshold states

In the above we have found that there are no bound states in the inverse power potential with a complex strength (including the case of the infinitesimal imaginary part). The physical reason is the absence of reflected wave from the absorptive core of the inverse power potential. In this subsection we study how the spectrum of the near-threshold states of a *regular* potential $U(r)$ (*i.e.* a potential which increases in the origin slower than $1/r^2$) is modified by a potential which has inverse power behavior $-(\alpha_s + i0)/r^s$ near the origin. The cases when such a modification is small are of special interest to us.

2.3.1 Penetration through the centrifugal barrier

It can be expected that the effect of the inverse power core could be small when the (regularized) inverse power potential is separated from the regular one by the centrifugal barrier. In this case the shift and width of the corresponding states would be determined by the centrifugal-barrier penetration probability. In fact, if α_s is small enough, there is a range of r where

$$U(r) \ll \alpha_s/r^s \ll (l(l+1) - \alpha_2)/r^2\,. \qquad (16)$$

Let us suggest that the regular potential is approximately constant in the mentioned range of r, so that $U(r) \approx p_0^2$. Then from (16) we get

$$p_0 \alpha_s^{1/(s-2)} \ll 1 \,. \tag{17}$$

For such values of r the wave function is

$$\Phi \sim \sqrt{r}(J_\nu(p_0 r) - \tan(\delta_s) Y_\nu(p_0 r)) \,, \tag{18}$$

$$Y_\nu = \frac{J_\nu \cos(\nu\pi) - J_{-\nu}}{\sin(\nu\pi)} \,, \tag{19}$$

$$\nu = \sqrt{(l+1/2)^2 - \alpha_2} \,,$$

here δ_s is a short-range contribution to the phase shift, produced by the inverse power potential $-(\alpha_s + i0)/r^s$ in the presence of the regular potential $U(r)$.

For small $r \sim \alpha_s^{1/(s-2)}$ the wave function is determined by the regularized inverse power and the centrifugal potential:

$$\Phi \sim \sqrt{r} H_\eta^{(1)}\left(\frac{2\sqrt{\alpha_s}}{s-2} r^{-(s-2)/2}\right) \,,$$

$$\eta = 2\nu/(s-2) \,.$$

Matching the logarithmic derivatives and taking into account (17), we get for the phase shift

$$\delta_s = -B_\nu \left(\frac{p_0 \alpha_s^{1/(s-2)}}{2(s-2)^{2/(s-2)}}\right)^{2\nu} \exp(-i\pi\eta) \,,$$

where

$$B_\nu = \sin(\pi\nu) \frac{\Gamma(1-\nu)\,\Gamma(1-\eta)}{\Gamma(1+\nu)\,\Gamma(1+\eta)} \,.$$

The above-presented expression for $\mathrm{Re}\,\delta_s$ is accurate, strictly speaking, only in case $2l + 3 < s$ [32]. The correction to the real part of the phase shift which comes from large distances can be calculated as the first order of a distorted-wave approximation with respect to the inverse power potential.

At the same time δ_s has the positive imaginary part according to the "inelastic" character of regularized inverse power potential:

$$\mathrm{Im}\,\delta_s = (-1)^l \left(\frac{p_0 \alpha_s^{1/(s-2)}}{2(s-2)^{2/(s-2)}}\right)^{2l+1} B_\nu \,. \tag{20}$$

The near-threshold states produced by the regular potential $U(r)$ are perturbed by the regularized (absorptive) inverse power potential. In particular they get the widths, which in our case of small δ_s are proportional to $\mathrm{Im}\,\delta_s$.

If the near-threshold states spectrum in $U(r)$ has a semiclassical character, then from the quantization rule

$$\int \sqrt{E_n + \delta E_n - U(r)}\,\mathrm{d}r + \delta_s = \mathrm{const}$$

we get

$$\delta E_n = -\delta_s \omega_n \,, \tag{21}$$

where ω_n is a semiclassical frequency:

$$\omega_n = \left(\int (E_n - U(r))^{-1/2} \mathrm{d}r\right)^{-1} \,.$$

Taking into account (20), we get for the width of the state

$$\Gamma_n/2 = (-1)^{l+1} \left(\frac{p_0 \alpha_s^{1/(s-2)}}{2(s-2)^{2/(s-2)}}\right)^{2l+1} B_\nu \omega_n \,.$$

Thus in the above-mentioned case the modification of the near-threshold spectrum of the regular potential $U(r)$ by the regularized inverse power potential results in shifting and inelastic broadening determined by the small parameter $(p_0 \alpha_s^{1/(s-2)})^{2l+1}$, which characterizes the centrifugal-barrier penetration probability.

2.3.2 Quantum reflection states

We will treat here an interesting case, *i.e.* when there is no barrier separation between the regularized (absorptive) inverse power potential $-(\alpha_s + i0)/r^s$ and the regular potential $U(r)$. However, the existence of the narrow near-threshold states is still possible. The reason why in such a case rather narrow states can survive is the already mentioned quantum reflection from those parts of *attractive* potential, which change sufficiently fast.

To illustrate this idea, let us suppose that the full interaction potential $W(r)$ has the following simple form:

$$W(r) = \begin{cases} -(\alpha_s + i0)/r^s & \text{if } r < R_0 \,, \\ -(\alpha_s/R_0^s)\Theta(R-r) & \text{if } r \geq R_0 \,. \end{cases} \tag{22}$$

This potential can be treated as a shallow and wide square well with depth α_s/R_0^s and width R modified near the origin by the regularized inverse power potential $-(\alpha_s + i0)/r^s$, which is matched with the above-mentioned square well at distance R_0. We are interested to verify whether the existence of narrow states in such a potential is possible. For the moment we will restrict our treatment to the S-wave case only.

Let us suppose that $R_0 > r_{sc} \equiv (2\sqrt{\alpha_s}/s)^{2/(s-2)}$. Previously we showed that the WKB approximation, applied to the zero energy scattering on the regularized singular potential fails for $r > r_{sc}$. Thus in our problem there is a domain $r_{sc} < r \leq R_0$ of WKB failure. We will show that this domain acts similar to the barrier in the sense that it is responsible for the reflected wave generation.

In the vicinity of R_0 the zero energy wave function in regularized inverse power potential has the form

$$\Phi_1(r) \sim 1 - r/a \,.$$

Here a is the *complex* scattering length (9) in the regularized inverse power potential $-(\alpha_s + i0)/r^s$. The wave function in the square-well potential is

$$\Phi_2(r) \sim \sin(k_n r + \varphi_n) \,.$$

Here $k_n^2 = \alpha_s/R_0^s - E_n$, where E_n is the energy of the near-threshold state, while φ_n is the phase-shift to be determined. Matching the logarithmic derivatives at point R_0 and expecting that $|k_n R_0 + \varphi_n| \ll 1$, we find

$$\varphi_n = -k_n a\,.$$

To ensure the existence of the near-threshold state of interest (such that $|k_n(R_0 - a)| \ll 1$), the width of the square well R should be big enough. The required characteristic value R_c can be obtained from the condition of the state appearance in the square well of the depth α_s/R_0^s:

$$\sqrt{\alpha_s/R_0^s}R_c = \pi/2\,.$$

Matching the logarithmic derivatives of the square well wave function and the decaying wave at point R, we get for the bound energy $E_n = -\kappa_n^2$:

$$k_n \cot(k_n(R - a)) = -\kappa_n\,,$$

which for $\kappa_n \ll k_n$ gives

$$k_n \simeq \sqrt{\alpha_s/R_0^s}\,, \tag{23}$$

$$\mathrm{Re}\,\kappa_n = k_n^2(R - R_c - \mathrm{Re}\,a)\,, \tag{24}$$

$$\mathrm{Im}\,\kappa_n = -\,k_n^2\,\mathrm{Im}\,a\,. \tag{25}$$

The corresponding S-matrix pole in the complex k-plane is $z = i\kappa_n$. Depending on the sign of $\mathrm{Re}\,\kappa_n$ the mentioned pole can be either in the upper half-plane, which corresponds to the bound state, or in the lower half-plane, which corresponds to the virtual state. The width of such a state is

$$\Gamma/2 = k_n^4\,\mathrm{Im}\,a(R - R_c - \mathrm{Re}\,a)\,.$$

The effect of the regularized singular potential is determined by the parameter $k_n\,\mathrm{Im}\,a$. The physical sense of such a parameter can be easily established. In fact, the S-matrix corresponding to the scattering with a small momentum k on the regularized inverse potential is

$$S = 1 - 2ika$$

The intensity of the reflected wave is

$$P = |S|^2 = 1 - 4k|\,\mathrm{Im}\,a|\,.$$

The smaller $k\,\mathrm{Im}\,a$ is the higher is the probability of the quantum reflection and the less is the influence of the regularized singular potential on the near-threshold spectrum of the regular potential.

So far the narrow states can exist only if $k_n|\,\mathrm{Im}\,a| \ll 1$. If the square well is chosen so deep that $R_0 < r_{sc}$, then there is no domain of WKB failure, the probability of quantum reflection is negligible and no narrow states can be formed.

From the above treatment we can derive the estimation for the maximum binding energy of the narrow quasi-bound state in a regular potential $U(r)$ modified by a regularized inverse power potential.

Let R_{sc} be the distance where the WKB failure takes place for the full interaction $W(r) = U(r) - (\alpha_s + i\omega)/r^s$. As we have shown the effect of the WKB failure domain is the partial reflection. Thus, for the purpose of the qualitative estimation, we can replace this domain by the boundary condition of full reflection at R_{sc}. In other words, we should look for the bound states in the following truncated potential:

$$W_{tr}(r) = \begin{cases} +\infty & \text{if } r < R_{sc}\,, \\ W(r) & \text{if } r \geq R_{sc}\,. \end{cases} \tag{26}$$

The ground-state energy E_{tr} in such a potential gives an approximation for the lowest (quasi-)bound state energy in the full potential $W(r)$.

Summarizing the above results we can expect the existence of narrow quasi-bound state if the regular potential, responsible for the state formation, is separated from the regularized inverse power potential either by the centrifugal barrier or by the domain of WKB failure.

3 Optical model of the $N\overline{N}$ interaction

It is clear from the above results that the model potential, which behaves at short distance like $-(\alpha+i\omega)/r^3$ enables a unique solution of the scattering problem. Such a potential is absorptive and accounts not only for elastic, but for inelastic processes as well. The above statements are true even for infinitesimal value of ω. As we have shown such an infinitesimal imaginary addition is equivalent to the full absorption boundary condition in the origin.

We first apply this regularization procedure to the $N\overline{N}$ potential in the $^{13}P_0$ state. The $N\overline{N}$ interaction in this state includes singular terms. This particular state is also interesting due to the presence of the near-threshold resonances, predicted by most of the theoretical models. We choose the version of the real OBE potential from the Kohno-Weise model [7] accompanied by the imaginary component $-i\omega/r^3$ with $\omega \to 0$:

$$V = V_{\mathrm{KW}} - i\omega/r^3\,.$$

Here V_{KW} is the real part of the Kohno-Weise potential *without any cut-off.*

The scattering volume in the $^{13}P_0$ $T = 0$ state calculated in the limit $\omega \to 0$ turns out to be

$$a_r = -7.66 - i4.87\ \mathrm{fm}^3\,,$$

while the value obtained within the Kohno-Weise model with a cut-off $r_c = 1$ fm is

$$a_{\mathrm{KW}} = -8.83 - i4.45\ \mathrm{fm}^3\,.$$

As we can see, both scattering volumes are rather close. The above procedure can be successfully applied to any $N\overline{N}$ state which includes attractive singular terms.

An interesting question is wether it is possible to describe the *whole set* of low-energy scattering data within a model of OBE potential accompanied by the assumption

Table 1. S- and P-wave scattering lengths.

State	DR1	DR2	KW	Reg
$^{11}S_0$	$0.02-i1.12$	$0.1-i1.06$	$-0.03-i1.35$	$-0.08-i1.16$
$^{31}S_0$	$1.17-i0.51$	$1.2-i0.57$	$1.07-i0.62$	$1.05-i0.55$
$^{13}S_1$	$1.16-i0.46$	$1.16-i0.47$	$1.24-i0.63$	$1.19-i0.64$
$^{33}S_1$	$0.86-i0.63$	$0.87-i0.67$	$0.71-i0.76$	$0.7-i0.65$
$^{11}P_1$	$-3.33-i0.56$	$-3.28-i0.78$	$-3.36-i0.62$	$-3.19-i0.59$
$^{31}P_1$	$0.92-i0.5$	$1.02-i0.43$	$0.71-i0.47$	$0.81-i0.46$
$^{13}P_0$	$-9.58-i5.2$	$-8.53-i3.51$	$-8.83-i4.45$	$-7.67-i4.74$
$^{33}P_0$	$2.69-i0.13$	$2.67-i0.15$	$2.43-i0.11$	$2.46-i0.15$
$^{13}P_1$	$5.16-i0.08$	$5.14-i0.09$	$4.73-i0.08$	$4.75-i0.15$
$^{33}P_1$	$-2.08-i0.86$	$-2.02-i0.7$	$-2.17-i0.95$	$-2.09-i0.79$
$^{13}P_2$	$0.04-i0.57$	$0.22-i0.56$	$-0.03-i0.88$	$-0.12-i0.82$
$^{33}P_2$	$-0.1-i0.46$	$0.05-i0.55$	$-0.25-i0.39$	$-0.14-i0.39$

of full absorption of the particles in a small volume around the origin.

We suggest the following modification of the $N\overline{N}$ potential model:

$$\widetilde{V} = V_{\mathrm{KW}} - i\frac{A}{r^3}\exp(-r/\tau)\,. \qquad (27)$$

The parameters of the imaginary part of the potential were taken as follows: $A = 4.7$ GeV fm^2, $\tau = 0.4$ fm. We have calculated the values of S and P scattering lengths in such a model potential. The obtained results, (indicated as Reg) together with the results of two Dover-Richard models, (DR1 and DR2), and the Kohno-Weise model (KW) taken from [33] are presented in table 1. In this table the S-wave scattering lengths are given in fm, while the P-wave scattering volumes are given in fm^3.

One can see rather good agreement between the results obtained within the suggested optical model without cut-off and the cited above versions of Kohno-Weise and Dover-Richard models.

Let us underline here, that there are no reasons to believe that the physical interaction indeed has the above form at small distances. The physical sense of the suggested model is that the low-energy scattering observables are independent of the the short-range interaction details, as long as such interaction includes the strong inelastic component. These observables can be obtained from the solution of the Schrödinger equation, formally applied to the short distances.

3.1 Near-threshold resonances

The critical question of the quasi-nuclear model is whether the strong annihilation could be compatible with the existence of narrow quasi-bound $N\overline{N}$ states. It follows from the above-performed analysis of our regularized potential model that only few performed (if any) near-threshold quasi-bound states or resonances can survive, while any deeply bound states are excluded.

We examined the S-matrix poles in the $^{13}P_0$ state. The real and imaginary parts of the scattering volume in such a state as they appear in our calculations are rather large $a = -7.67 - i4.74$ fm^3. Such a big value of the scattering volume could be an indication of the near-threshold state or resonance.

In fact we found that the S-matrix poles nearest to the threshold are situated in the third and fourth quadrant of the complex k-plane:

$$k_+ = 44.8 - i54.3\,\mathrm{MeV}/c \quad k_- = -58.4 - i73.8\,\mathrm{MeV}/c\,.$$

In the absence of annihilation these poles should be symmetrical with respect to the imaginary axis and correspond to the near-threshold resonance. The short-range annihilation breaks the left-right symmetry between such poles. Such near-threshold poles will manifest themselves by a rapid increasing of the related amplitude and the cross-section with the energy decreasing down to the threshold.

The found resonance belongs to the above-mentioned "quantum reflection" case. Indeed, the analysis of the $N\overline{N}$ potential (27) (into which the centrifugal term is included) in the $^{13}P_0$ state shows that the centrifugal barrier is overcome by the attractive singular terms. Thus there is no barrier between the absorptive core and the regular part of the interaction. It was found that the WKB failure condition takes place for $r \geq R_{sc} = 1.4$ fm. This distance plays the role of effective annihilation radius in the $^{13}P_0$ state. To judge about the spectrum, we examine the potential truncated at point $R_{sc} \simeq 1.4$ fm (26). One can check that such a potential supports no bound states and the S-matrix pole nearest to the threshold indeed corresponds to the resonance.

Thus the P-wave enhancement is explained in the presented model by the existence of the near-threshold resonance. It should be specially mentioned that no deep bound states with given quantum numbers could exist in our model in spite of the strong $N\overline{N}$ attraction.

It is worth mentioning that the suggested regularizing procedure can be applied to any OBEP-based optical model of the $N\overline{N}$ low-energy interaction.

4 Conclusion

We have found that scattering observables are insensitive to the details of the short-range interaction, if such an interaction includes a strong inelastic component. The corresponding scattering amplitudes can be derived from the solution of the Schrödinger equation, formally applied to the short distances. Mathematically this means that the singular attractive terms of the $N\overline{N}$ potential can be regularized by an imaginary addition to the interaction strength. We analyzed the main properties of such a regularization. In particular it was found that the inverse power potential regularized in the mentioned way supports no bound states. The low-energy scattering amplitude on such a potential is determined by the quantum reflection from the region, where the WKB approximation fails (the potential tail). The mentioned formalism was used to build an optical model of the $N\overline{N}$ low-energy interaction free from uncertainty, related to the cut-off parameter. The good agreement with the results, obtained within well-established optical models, was demonstrated. We proved that no deep quasi-bound states were possible within our model. The spectrum of narrow quasi-nuclear states is concentrated near the threshold. In particular, we demonstrated the existence of the near-threshold resonance in the $^{13}P_0$ state, responsible for the P-wave enhancement. It is argued that the existence of such resonances is possible due to the phenomenon of quantum reflection from those parts of $N\overline{N}$ potential which change sufficiently fast. These domains of the WKB failure play a role of effective barrier between the regular part of the interaction and the absorptive singular core.

The research was performed under the support of the Russian Foundation for Basic Research grant 02-02-16809.

References

1. R.J.N. Phillips, Rev. Mod. Phys. **39**, 681 (1967).
2. I.S. Shapiro, Phys. Rep. C **35**, 129 (1978).
3. O.D. Dalkarov, F. Myhrer, Nuovo Cimento A **40**, 152 (1977).
4. T. Ueda, Prog. Theor. Phys. **62**, 1670 (1979).
5. J. Cote, M. Lacombe, B. Loiseau, B. Moussallam, R. Vinh Mau, Phys. Rev. Lett. **48**, 1319 (1982).
6. J. Carbonell, O. Dalkarov, K. Protasov, I. Shapiro, Nucl. Phys. A **535**, 651 (1991).
7. M. Kohno, W. Weise, Nucl. Phys. A **454**, 429 (1986).
8. C.B. Dover, J.-M. Richard, Phys. Rev. C **21**, 1466 (1980).
9. M. Pignone, M. Lacombe, B. Loiseau, R. Vinh Mau, Phys. Rev. C **50**, 2710 (1994).
10. B. El-Bennich, M. Lacombe, B. Loiseau, R. Vinh Mau, Phys. Rev. C **59**, 2313 (1999).
11. W.W. Buck, C.B. Dover, J.-M. Richard, Ann. Phys. (N.Y.) **121**, 47 (1979).
12. A. Martin, *Antinucleon-Nucleon Scattering at LEAR, Proceedings of the 8th Winter Course, Folgaria, Italy*, edited by M. Gibilisco, G. Preparata, A. Zenoni (World Scientific, Singapore, 1994).
13. C. Amsler *et al.*, Phys. Lett. B **297**, 214 (1992).
14. A. Abele *et al.*, Eur. Phys. J. C **17**, 583 (2000).
15. D. Gotta *et al.*, Nucl. Phys. A **660**, 283 (1999).
16. M. Augsburger *et al.*, Nucl.Phys. A **658**, 149 (1999).
17. F. Iazzi *et al.*, Phys. Lett B **475**, 378 (2000).
18. A. Bianconi *et al.*, Phys. Lett B **483**, 353 (2000).
19. F. Nichitiu *et al.*, Phys. Lett B **545**, 261 (2002).
20. P. Montagna *et al.*, Nucl. Phys. A **700**, 159 (2002).
21. T. Walcher, Annu. Rev. Nucl. Part. Sci. **38**, 67 (1988).
22. E. Klempt, F. Bradamante, A. Martin, J.-M. Richard, Phys. Rep. **368**, 119 (2002).
23. J.Z. Bai *et al.*, Phys. Rev. Lett. **91**, 022001 (2003)
24. L.D. Landau, E.M. Lifshitz, *Quantum Mechanics* (Pergamon Press, New York, 1977).
25. J. Carbonell, Rev. Mex. Fis. **47**, 70 (2001); J. Carbonell, M. Mangin-Brinet, *Selected topics on Low Energy Antiproton Physics*, e-print arXiv: nucl-th/0103043.
26. A.Yu. Voronin, Phys. Rev. A **67**, 062706 (2003)
27. M.F. Mott, H.S.W. Massey, *The Theory of Atomic Collisions* (The Clarendon Press, Oxford, 1965).
28. G.N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, 1922).
29. J.E. Lennard-Jones, Proc. R. Soc. London, Ser. A **156**, 6 (1936).
30. R. Cote, H. Friedrich, J. Trost, Phys. Rev. A **56**, 1781 (1997).
31. H.Friedrich, G. Jacoby, C.G. Meister, Phys. Rev. A **65**, 032902.
32. V.V. Babikov, *The Method of Phase Functions in Quantum Mechanics* (Nauka, Moscow, 1967) (in Russian).
33. J. Carbonell, J.M. Richard, S. Wycech, Z. Phys. A **343**, 325 (1992).